THE RESEARCH AS A PRINCIPLE FOR THE TEACHING AND LEARNING OF MATHEMATICS

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ABSTRACT

In this article I present some notes that reflect on some changes in the educational process at school, taking them as support to argue about the possibility of including research as a principle of teaching and learning. To this end, I chose the training of teachers who teach mathematics as the focus of my discussion, because this training in teaching (Mathematics and Pedagogy) must be founded on epistemological bases which suppose the investigating spirit as a learning agent of the mathematics addressed in these academic courses. The article has been divided into six parts. In the first one I present the initial assumptions on the subject of research and knowledge production. The second one refers to the epistemological foundations of research in the production of knowledge as support so that in the third part, I might be able to address the problem solving, imagination, elaboration and representation of knowledge. Then I discuss the didactical aspects related to research as a teaching and learning mathematical principle. Finally I describe some experiences in the training of teachers who teach mathematics and point out possibilities of a teacher training for research in the teaching degrees focused on in this article.

Keywords: Research. Teaching and learning of mathematics. Mathematical knowledge production. Mathematics teachers training. Mathematics education.

RESUMO

Neste artigo vou discutir a possibilidade de incluir a pesquisa como um princípio de ensino e aprendizagem. Para este fim, eu escolhi a formação de professores que ensinam matemática como o foco da minha discussão, porque essa formação em ensino (Matemática e Pedagogia) deve ser fundada em bases epistemológicas que supõem o espírito investigativo como um agente de aprendizagem da matemática. O artigo foi dividido em seis partes. Na primeira, apresento os pressupostos iniciais sobre o tema da produção de conhecimento e pesquisa. A segunda refere-se aos fundamentos epistemológicos da pesquisa na produção do conhecimento como suporte para que na terceira parte, seja possível lidar com a resolução de problemas, imaginação, elaboração e representação do conhecimento. Discutirei, desse modo, os aspectos didáticos relacionados com a pesquisa como um princípio de ensino e aprendizagem da Matemática. Finalmente descrevo algumas experiências na formação de professores que ensinam matemática e aponto possibilidades de uma formação de professores para a investigação nos graus de ensino focados neste artigo.

Palavras-chave: Pesquisa. Ensino e aprendizagem da matemática. Produção de conhecimento matemático. Formação de professores de Matemática. Educação matemática.

Presentation

Throughout its history the university has always suffered changes and adaptations in the teaching and researching process, caused often by the interactions between science and the social, cultural and political contexts. This movement through which science has gone and is still going today, involves changes in the educational process established in the school and proposed to the society. In this article I present some notes that reflect on some changes in the educational process at school, taking them as support to argue about the possibility of including research as a principle of teaching and learning. To this end, I chose the training of teachers who teach mathematics as the focus of my discussion, because this training in teaching (Mathematics and Pedagogy) must be founded on epistemological bases which suppose the investigating spirit as a learning agent of the mathematics addressed in these academic courses.

In this sense, I consider that the teaching of mathematics in the university context must focus on three aspects: society, culture and cognitive process – generated by and in the former two, which include the produced mathematics. The thesis proposed here is that the training of teachers who teach mathematics should take the exercise of investigation as a teaching and learning principle that can train professionals capable of making their teaching practice a constant coming and going in the development of students' cognition of Middle and High School. Thus it might be possible to contribute to the formation of autonomous individuals to solve problems encountered in their personal and professional actions. My exposure hovers between the paradigmatic and pragmatic axes of the training of teachers who teach mathematics.

This article has been divided into six parts. In the first one I present the initial assumptions on the subject of research and knowledge production. The second one refers to the epistemological foundations of research in the production of knowledge as support so that in the third part, I might be able to address the problem solving, imagination, elaboration and representation of knowledge. Then I discuss the didactical aspects related to research as a teaching and learning mathematical principle. Finally I describe some experiences in the training of teachers who teach mathematics and point out possibilities of a teacher training for research in the teaching degrees focused on in this article.

Initial assumptions

It is known that since prehistoric times mankind develops cognitive strategies with prospects of reading, interpreting, comprehending and explaining the natural, social and cultural realities concerning its survival on the planet. In the course of their socio-historical and cultural development, societies have always sought to build spaces that ensured the exchange of cognitive strategies which had been generated, as well as the consolidation and dissemination of knowledge gained from the exercise of these strategies. Historically, the production process of human knowledge became a reality with the formation of a *corpus* of theoretical and practical know-how which preserved the thinking strategies generated in different socio-cultural contexts, as well as its expansion in multiple dimensions, to find solutions to problems of human survival.

The environments for the exchange of the produced knowledge have materialized with the creation of the school environments, on different levels and patterns, as the historical, philosophical and cultural context of each society. In this context, as Wilhelm Von Humboldt (1997) ensures, the model of university education appeared with a view to create environments of formalization, systematization and validation of knowledge, often produced in different socio-cultural contexts. From this movement emerged, mainly, philosophical currents which delineate the epistemological conceptions about knowledge, science and education, considering the university as the main vehicle of teaching, research, and dissemination of the production of science, technology and professional training.

University, however, has changed over time with respect to such procedural pillars on which it relies for its existence along its historical path. Such changes might be caused by the need of establishing a socio-cognitive and cultural dialogue between everyday knowledge, educational knowledge, and scientific knowledge in order to respond to the various problematic issues arising in the context of society and culture, so that the solutions are systematized socially disseminated, reformulated, and so forth.

This is a cyclical, ondulatory process in which the produced knowledge is characterized by upper and lower peaks of validation, according to the contextual references used to visualize the trajectory of our cognitive creation. In this regard, in 1934, Karl Popper argued that it was neither possible nor necessary to justify the laws of science by way of the justification of inductive reasoning, asserting that scientific theories are not derived inductively from facts. Initially, hypotheses and speculations are created to then be subjected to experimental tests and criticisms in an attempt to refute or validate the results and conclusions. This is the book published in English in 1959 under the title *The Logic of Scientific Discovery*, later translated and published in Portuguese as *A Lógica do Descobrimento Científico*.

What Popper postulated was that every theory should be available for testing and for the risk of being rejected, involving the achievement or not of credibility in the scientific community. Such an assertion means that a scientific theory can be objectively true, but we can never know it with certainty (cf. LAKATOS, 1998). Even with all the changes in the science scene in the twentieth century, when it comes to mathematics, there was a somewhat stagnation. Discussions between formalists, intuitionists and logicists originated a program of mathematical research which contributed solely to the body of mathematics itself, having stagnated or disappeared later.

The result was the need of investigating mathematics in an attempt to review the uncertainties observed by the mathematicians themselves concerning the nature, purpose and importance of mathematical research. In this expectation Imre Lakatos proposed his theory on the *logic of mathematical discovery*, based on two theoretical pillars: the science philosophy of Karl Popper and the heuristics of mathematical discovery proposed by **George Polya (1962),** based on the analysis of arguments and counter arguments about the mathematical problematization (See more details in the book *Proofs and refutations: the logic of mathematical Discovery*, later translated and published in Portuguese under the title *A lógica do descobrimento matemático: provas e refutações*, published in 1978 by Publisher Zahar). This was the basis of his thesis on the *logic of mathematical discovery*, based on the dialogue between the *evidence and refutations* of a mathematical proof. Lakatos's work can be considered a decisive contribution to the further implementation of mathematical investigation in the classroom as a didactical strategy in Mathematics Education.

In this sense, his constant search for explanatory answers about the validity of mathematical knowledge produced throughout history is made evident when Lakatos (1981, p. 237), assures us that

> (...) "Due to the justificationist losses so deeply entrenched against refuted theories, scientists often minimize the refuting instances and do not take seriously into account a falsifying hypothesis before the latter is included in a rival theory of higher level which explains, moreover, the partial success of the refuted theory. Until that moment arrives, the falsifying hypotheses are excluded by the public body of science. However, it also happens of a theory being publicly refuted even if it has not been replaced by another: its falsehood known, the theory continues to be explained and contrasted. In such cases the theory is officially registered as a theory that in its current version, applies only to *ideal* or *normal* cases, etc ... and its falsifying hypotheses, when they are mentioned, register as *anomalies*".

Based on this argument from Lakatos it is possible to admit that the centered standards of validation and enhancement of scientific creation and dissemination, especially through the universities, from social patterns, culturally and politically existent, result, among other products, in the transformation of the ways of understanding and making education. This transformation concerns the ways and means of setting up educational training in all its levels. Among them is college education, especially teacher training in all areas of science education coverage.

Epistemological foundations of research in knowledge production

As mentioned in the last paragraphs of the previous section, the exercise of validation and appreciation of scientific creation and diffusion has generated over the past centuries, a series of discussions and debates among philosophers, scientists and epistemologists in order to understand and explain the most appropriate methods and models for what could be considered as scientific knowledge.

During the late nineteenth century and first half of the twentieth century, some situations were adjusted concerning the establishment of branches for the sciences, fed by a group of philosophers and scientists amongst which stood out Karl Popper, Gaston Bachelard, Thomas Kuhn, Rudolf Carnap, Imre Lakatos, among others who constitute the philosophical and epistemological foundations of Western science in the twentieth century (Cf. Gonçalves-Maia, 2011).

One aspect which has a direct relationship with the supporting bases of the didactical prospect referring to research and teaching as a principle of teaching and learning, the central focus of this conference, emerges from the discussions proposed by Lakatos (1978, 1981), as he was proposing his investigating principles. In describing and arguing about his *methodology of scientific research programs*, Lakatos (1981, 1998) developed a description of science as an attempt to improve the Popperian falsificationism and overcome the objections to it. In this regard Chalmers (1993, p. 118) ensures that

> “the fact that any part of a complex theoretical maze which may be held responsible for an apparent falsehood poses a serious problem for the falsifier who relies on an unconditional method of conjectures and refutations. For them, the inability to locate the source of the problem resulted in a non-methodical chaos. Lakatos' account of science is sufficiently structured to avoid that consequence. Order is maintained by the inviolability of the irreducible core of a program and by the positive heuristic which accompanies it. The proliferation of ingenious conjectures within this framework will lead to progress, with the condition that some of the resulting predictions stemming from the ingenious conjectures occasionally prove to be successful. The decisions of withholding, rejecting a hypothesis are determined quite directly by the results of experimental tests. Those who survive the experimental tests are retained, although some decisions are open to appeal, in light of some ingenious further hypothesis, independently testable. The importance of an observation for an hypothesis being tested is not as problematic within a research program, for the irreducible core and the positive heuristic serve to define a language of rather stable observation”.

In *The logic of mathematical discovery: evidence and refutations*, Lakatos (1978) presents a valuable theoretical-practical contribution so that later some researchers in Mathematical Education could envision and establish guidelines of a didactical proposal focused on the teaching of mathematics research in the classroom. (See details in Ponte, Brocardo and Oliveira, 2003; Mendes, 2009a). In this work, Lakatos uses history as a basis to argue that as the natural sciences, mathematics is fallible and arguable, that is, it also develops through the criticism and correction of theories which are never entirely free of ambiguity.

In exemplification for his argument, Lakatos presents a model about the logic of mathematical discovery by building a dialogue between a teacher and their class when they are studying the famous Euler-Descartes formula for polyhedra: $V + F = A + 2$, where V, F and A represent the number of vertices, faces and edges of a polyhedron, respectively. The pedagogical-didactic dialogue established by Lakatos highlights speculative investigative actions conducted in the classroom in the sense of a group asking themselves about the process of research and discovery in mathematics in the search for explanations and solutions to the presented problematic, where all members of the group behave as researchers. This type of group organization oriented to experience an inquiring practice which can illustrate and lead to the comprehension of the arguments and solutions which include the construction and validation of the solutions to the discussed problems, and these groups are now called by some authors as communities of practice.

The conceptual category *Community of Practice* refers to a group of people who come together around a common topic or interest and work together seeking to find ways of improving what they do, ie in the resolution of a problem in the community or in daily learning through regular interaction. The term was coined by Jean Lave (1988) and extended in partnership as Etienne Wenger **(See Leave and Wenger, 1991)**. The word practice is not used here, as opposed to *theory,* not meaning a reflexive action either. The term *theory* also does not mean a thought without action (cf. Wenger, 2001, p. 71-72). The practices are not, then, designed as a set of actions synonymous to *activity*, although they may be conducted in different contexts of human activity. More details in this aspect are present in Miguel and Mendes (2010).

In *The logic of mathematical discovery: evidence and refutations* (Lakatos, 1978), we can see how the dynamics established between *conjectures, examples, counterexamples, formulation and reformulation of theorems,* develops in the classroom to the extent that discussions and arguments adjust themselves in the interaction between the subjects involved in the construction of the theory which is intended to teach and learn in school. This heuristic example given by Lakatos on the demonstrations and refutations was applied to the development of mathematical knowledge in general and was reformulated for the mathematical culture in general, implying the possibility of being used individually in an attempt to create new mathematics, as shown in the simplified model in figure 1, below.

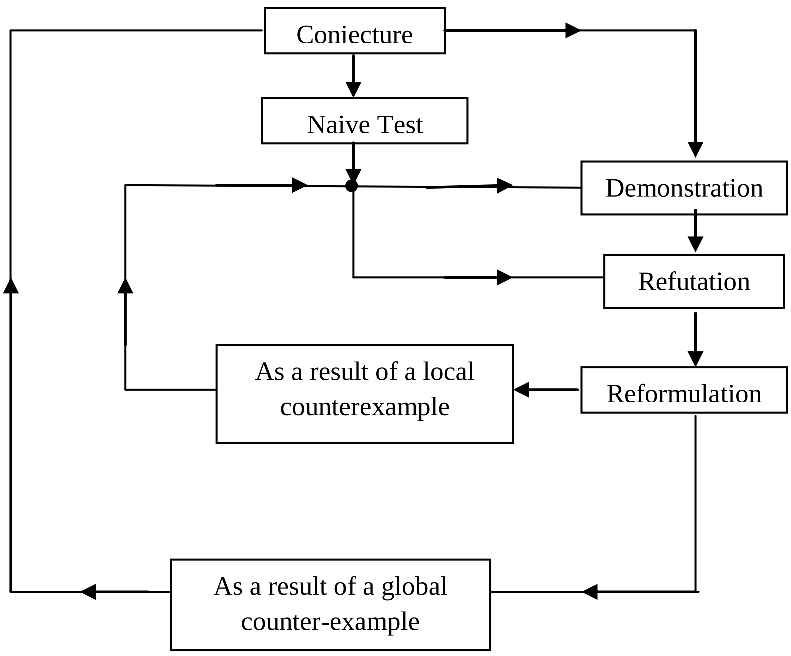

Figure 1. Simplified model from Lakatos for the heuristic of mathematical discovery.
(Cf. Davis and Hersh, 1995, p. 276).

According to Davis and Hersh (1995, p. 325), Lakatos did not apply this epistemological analysis to formal mathematics, but to the *informal* mathematics, based on the discovery and development process which is the culture of mathematicians and mathematics students. In fact, formalized mathematics, to which most of the recent philosophy is dedicated, it is virtually impossible to be found wherever we seek outside of texts and journals of symbolic logic.

This model was successfully used in many mathematics classrooms, although the initial shock of the students always happened when they were not presented with a fixed problem to be investigated and solved, but a problematic situation with open questions that should and could be overcome by them.

In *Criticism and the development of knowledge*, Lakatos and Musgrave (1979) claim that

> “There is an important demarcation between the "passivist" and "activist" theories of knowledge. The "passivists" support that true knowledge is the mark imprinted by nature on a perfectly inert mind: mental activity can only result in bias and distortion. The most influential Passivist school is the classical empiricism. The "activists" claim that we can not read the book of nature without mental activity, without interpreting it in light of our expectations or theories. Now conservative activists contend that we are born with our basic expectations; with them we transform the world into 'our world' but then we have to live forever in the prison of our world. The idea that we live and die in the prison of our ‘conceptual frameworks’ was first developed by Kant, the pessimistic Kantians thought that the real world is forever unknowable because of that prison, while the optimistic Kantians thought that God created our reference concept in order to adjust it to the world. But the revolutionary activists believe that the conceptual frameworks can be developed and also replaced by new and better benchmarks, we are the ones who create our "prison" and we can also, critically, demolish them” (Lakatos and Musgrave, 1979, p. 126).

It is based on an approximate approach to revolutionary activism that I present my way of conceiving the production of mathematical knowledge through research, supporting myself in the pedagogical potential of projects championed by John Dewey (1859-1952) and William H. Kilpatrick (1871-1965) at the beginning of the twentieth century. The pedagogical principles of research in mathematics education, proposed here, have their epistemological bases present in *The logic of mathematical discovery: evidence and refutations* and in the pedagogy of projects by Kilpatrick and Dewey.

In his History of Pedagogy, Franco Cambi (1999) presents considerations for the inclusion of the experimental method in schools between the late nineteenth and early twentieth century. In this regard he mentions the influences exerted by the works of Johann Heinrich Pestalozzi, Maria Montessori, Jean-Ovide Decroly, Celestin Freinet, William Kilpatrick and John Dewey. It is the emergence of a movement called active school which lasted in some countries, including Brazil, for some years, disappearing for a period and returning to the educational scene in the late twentieth century.

Currently, the use of projects in the development of interdisciplinary activities in school, is taken as an educational concept generated by the works of Dewey and Kilpatrick and which more recently, with the prospect of the pedagogy of projects, has established itself as an educational and training activity by excellence. Hence the justification for the acquisition of investigative knowledge, skills and attitudes during educational training.

Another reason relates to current ideas about the contextualized nature of learning and that which is known today about the relationship between motivation and cognition or the resolution of problems in an environment of teamwork. An extremely important aspect has been included in this practice today: the search for relationships between the everyday, educational and scientific aspects of the knowledge which is taught and learned in school, which reflects the interdisciplinary and transversal character which must be given to school knowledge in a pedagogical approach which involves problem solving, imagination, design and knowledge representation.

Problematization, imagination, formulation and representation of knowledge

Society develops its strategies of thought and action in order to find solutions for its problems and build its knowledge to be communicated and disseminated in the social context. Both school and science make use of this same principle to constitute the so-called institutionalized knowledge, that is, the knowledge considered scientific which is intended to be widespread in the school environment.

The knowledge currently disseminated in school is originated in information past and present, in addition to suffering a redefinition according to the sociocultural contextualization which overlays the information to be targeted in school activities. This knowledge stems from problematizations established in the interaction between society, culture and cognition in an attempt to find solutions to the problematic situations arising in the context. The solutions are configured in two ways: unresolved issues and unanswered questions.

The solved questions arise from the answers to the emerged problematization. To the extent that such responses are coded with a view to its communication and also its use in the search for answers about similarly problematic situations, new questions (open questions) almost always are originated. The open questions, in their turn, constitute provocative sources for new studies, thus becoming in a cyclical process of knowledge production.

In this determination of answers for the questions raised in the everyday problems as well as in their coding, the solutions always allow the emergence of new questions about the problem, which needs to be better explained. The open questions, however, arise between the lines of each solved and codified issue, causing further studies, feeding the process of knowledge generation in a cyclical act of strategy production and symbolic or mental representations which underlie the formalized models of the generated knowledge (See **Mendes, 2002; Mendes, 2003**).

The questions answered are constituted, therefore, in the basis for the generation of new cognitive strategies to answer the open questions and explain the doubts arising and/or for

the already existent questions, in order to create new representations which encode the new solutions obtained. Insofar as the issues are encoded, they constantly generate new questions which assume the place of new open questions. This process proceeds continuously in teaching and learning of school knowledge through the development of investigative activities directed and monitored by the teacher.
The descriptor shown in Figure 2 suggests a cycle of questioning, imagination, design and representation of knowledge, from an investigative process in which knowledge production is shaped epistemologically implying guiding pedagogical principles for an education supported by the use of information stemming from the context of society and culture throughout history and which may constitute a didactic approach to the classroom.

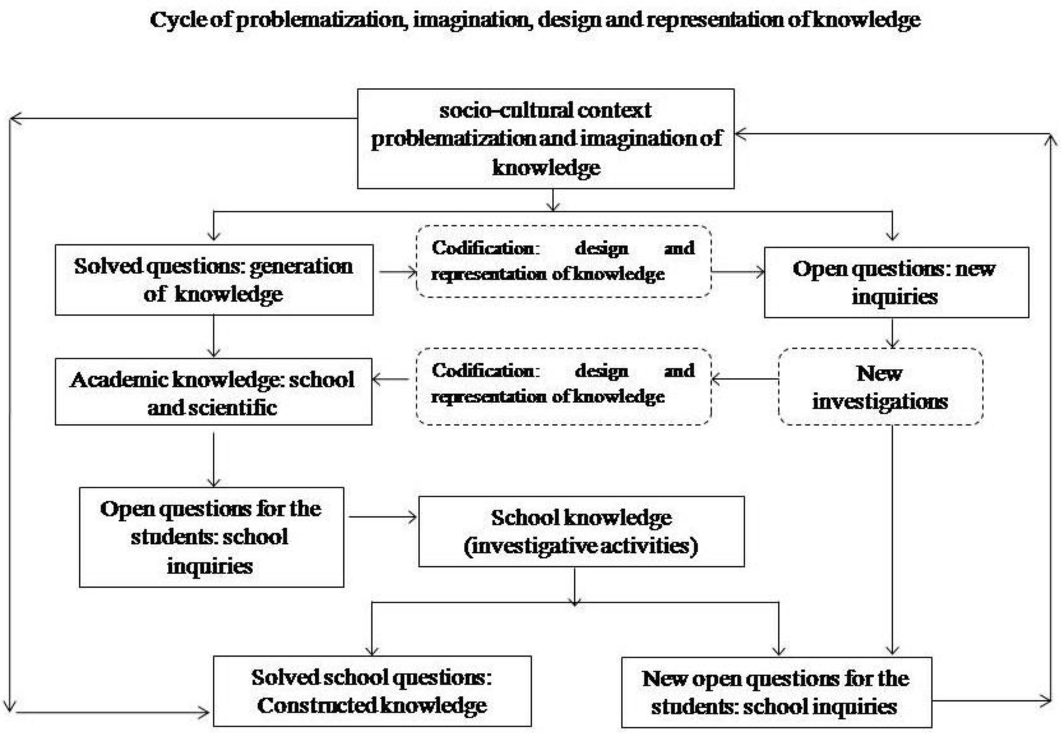

Figure 2. Cycle of Problematization, imagination, design and knowledge representation. Elaborated from **Mendes (2002, 2003)**

Based on the issues discussed and presented in the cycle described above, during the process of school knowledge construction, the information must be presented to the students in the format of open questions, mediated by the teacher in a learning process based on investigation. Such mediation can certainly raise questions to be solved by the students during the execution of investigative activities which will constitute the constructed knowledge, formalized and represented by them. However, this learning movement will give rise to other inquiries among students which will manifest themselves as new open issues to be investigated further by them, to expand their learning.

I wonder then: How to approach mathematics education, from this investigatory perspective? How to implement, in the classroom, this dynamic action to know mathematics? These were some of the concerns on which I relied for proposing didactical motions concerning the teaching of school mathematics, at different levels, when considering that my theoretical-practical model focused on historical research could help to overcome this problem (see Mendes, 2009a , 2009b).

The investigation is, then, a skill which marks our human characteristic in the world, defined by Teresa Vergani (2009) with the term *the creativity as destination*, since the investigative spirit constitutes the driving force of our reason for living and knowing, that is, our continuing search for answers for everything. This strategy might lead the student to a mental maturity that could make them more independent and aware of their ability to rely on curiosity and on the possibility of seeking knowledge through research. When this approach is based on historical mathematics information, it might be constituted as a generating source of mathematical school knowledge, that is, when undertaken in the classroom it might result in meaningful learning, and is materialized through activities focusing on the historical development of mathematical concepts (Cf. Mendes, 2009b).

The investigatory principles focused on finding solutions to issues emerging from the problematic contexts presented to the students will certainly contribute to that mathematics is viewed by them as science, language, game or art, since in them are interwoven four fundamental aspects of mathematics: the informative, the formative, the imaginative and the utilitarian, which formulate themselves amongst each other while being (re)used by society, school and the various cultural contexts. This is because insofar as the mathematical knowledge is deposited in the multiple experiences lived by the social groups, patterns arise, interpretive analogies and convergences (cognitive functions generated in the mind and by the mind of the learner human subject) which suggest the configuration of a generalizing language in which mathematics is to be structured. This language is characterized by the axiomatic presentation attributed to mathematics. The term learner is used by Hugo Assmann in the book *Re-enchant education*, to characterize the cognitive agent (individual, group, organization, institution, system) who is in active process of learning (Assmann, 1998, p.129).

Research as a principle of mathematical teaching and learning

The history of knowledge and social practices which are on the origin of this generalizing language is a strong evidence that mathematics is formulated in the analysis of the succession of difficulties found in different human contexts, leading to the arising of issues which cause the emergence of mathematical notions, concepts, definitions and properties.

As I've already mentioned in previous sections, the mathematician gets involved with a problem, in search for conjectures, seeking solutions for the problem, they analyze the solutions discovered in order to refute those which prove inadequate to the faithful solution of the investigated problem, demonstrate the process of searching solutions to this problem and rearranges the knowledge determined during the entire process (Cf. Bruter, 2000).

It is in pursuit of these rearrangements that the mathematician starts a new process: depersonalization, and decontextualization and the detemporalization of the produced knowledge, i.e. the elaboration of a general theory which represents the modeling of the mathematics practiced in search of solutions to a problem. Then begins the process of using the theory which was elaborated by others and, consequently, its subsequent (re)validation. According to the discovered solution, there may be a recast, application, generalization and the emergence of new conjectures stemming from the use of knowledge by others (cf. Brousseau, 2000). What to do so that this process arrives in the school in the form of systematic knowledge (school knowledge)?

The approaches to be executed in the context of university education must have as main goal to promote the acquisition of scientific education by the students. It is necessary, however, that the teachers propose and carry out training activities permeated by didactical strategies which stimulate the students' investigative spirit so as to articulate research with the training of future researcher-teachers (in the case of degrees) who make their teaching practice a constant coming and going in search for solutions to the students' cognitive development.

The investigative spirit of these students is reflected in the acquisition of a scientific attitude such as the formulation of hypothesis, demonstrations, model building, elaboration and mastery of mathematical language, concepts, theories, discussion of mathematical concepts, recognition of the various ways to represent the same concept according to the culture or context and the perception of the possibilities of standardization through the process of mathematical modeling, which can be enriched with the execution of investigative activities (research as a teaching method) which leads them to read, comprehend and formalize the investigated mathematics, in the form of concepts, properties and theories, etc.

The teacher's work resembles the work of the researcher (investigator), since he has to produce a (re)contextualization of knowledge so as to become learnable by the student. Thus, knowledge must emerge from investigation and adaptation to a particular situation, because the several topics that are to be taught do not create themselves under the same gender of context and relationship with the environment in which they are invented or used such as, for example, the arithmetic and algebra taught at school.

As Miguel and Mendes (2010) propose to us, the teacher must, therefore, simulate, in their classroom, a micro-scientific society or community of practice, if they want knowledge to be generated from good questions and debates, and encourage the creation of a language suitable to the demonstration of the questions' solutions generated and discussed in the classroom. The teacher should, therefore, propose and promote the creation of an investigative environment and investigative activities among the students.

It is in the construction of an investigative environment that the teacher might allow the students the development of mathematical skills to (re)decontextualize and (re)depersonalize their knowledge (know-how) from that which is presented to him by the scientific and cultural community of each time. It is, however, an exercise of scientific investigation with didactical purposes, since it has the student's mathematical learning as its main target. However, there is a guiding principle of this learning which is decisively formative: research as an educational principle.

In this regard, Alan Bishop (1999, p. 149), while arguing about the process of mathematical enculturation in the school curriculum mentions the need to include in this enculturation process to be effected by and in the school system, a essentially human component: mathematics as human culture and its relations with mathematical abstractions and its invention. Bishop also states that the school's mathematical curriculum should base itself on investigations in order to explain and comprehend the social and cultural aspects that underlie mathematics.

Also Emmánuel Lizcano (2009), in *Imaginário colectivo y creación matemática*, argues that the relations established between the social context, the creation and the implementation of valid practices in this context, concur to the production of mathematical knowledge. The author claims that mathematical conceptualizations emerge contaminated by the collective imaginary meanings which characterize the model of rational thought of every age and every culture. The author ensures that the socio-historical and cultural investigation leads us in search of answers to questions like: How each society builds the boundary which separates or unites possible and impossible, real and imaginary, thinkable and unthinkable, true and false?

To Bishop (1999), then a research project is an extensive work which can be done individually or in small groups, with the expectation of mimicking some of the activities of mathematicians, based on two distinct and complementary phases: 1) the creative and inventive phase characterized by exploration, analysis and development of mathematical ideas, 2) the writing of an account of activities undertaken in the first phase, i.e., reflection and communication on the theoretical formulations originated from the experiment executed in the first phase.

My approach to this issue resembles the considerations of Bishop and Lizcano in the sense of contributing to the reformation of thought about Mathematics Education, for insofar as we perceive the effectiveness of research in provoking the students' curiosity in understanding the development of mathematics and its socio-cultural origins, we understand its significance and its formalization as school mathematics. This integration results in a meaningful learning which operates itself in the investigatory activities, such as I present in this article.

I assure that mathematical research is full of surprises and that is so because it follows a method which is very close to the experimental method: the mathematician, as does the physicist, ventures a guess, observes the first result of an empirical generalization, and then tries to check it. If the mathematical invention, when scrutinized closely, is hesitant and is part of a method similar to the experimental one. In this sense, the use of research in the classroom can be driven directly or indirectly by the teacher, according to the methodological objectives and procedures set out in their educational planning.

Such didactical approach might be fulfilled according to stimulating and psychological environmental conditions of the student's and pedagogical ones of the teacher. Besides the aforementioned elements (objectives and procedures) I consider essential that the use of this approach might connect itself, as best as possible, to the stages of teaching development of each mathematical topic to be learned by the students.

To broaden the discussions already started, I denominate as investigative teaching activity the didactical decision given to every exercise of mathematical school knowledge generation which provokes the creativity and the challenging spirit of the students in an attempt to find answers to the cognitive questions caused by problematizations proposed by the teacher aiming to enable the construction of learning by the student. These research activities should constitute an ongoing process of knowledge construction, considering the connections between three essential components of a mathematical activity: intuitive, algorithmic and formal, as proposed by Efrain Fischbein (1987). So, we invest on the hypothesis that with this type of activity students might expand their ability of imagination, creativity and understanding of the mathematical aspects extracted from investigated situations as shown in Figure 3, below.

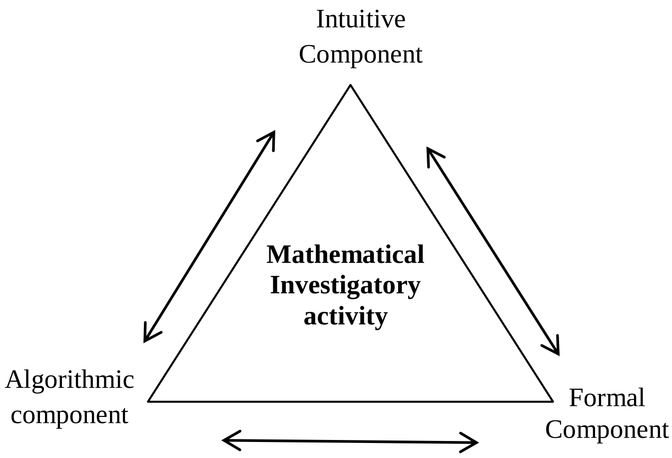

Figure 3. Descriptive of the relationship between the components of an investigative mathematical activity according to the elaboration of Mendes (2001), from the propositions of Efrain Fischbein (1987).

In this sense, the intuitive component manifests itself, for example, self-evident cognitions in the mind from learners. It is the creative imagination, visual interpretation, of the material explanation of an observed mathematical fact, experienced or imagined by the learner. The algorithmic component, however, constitutes itself in the exercise of organization skills and systematization of the creative imagination established by the intuition and which is put to the test on experimentation. The algorithms play an important role in the organization of mathematical reasoning by forming themselves into an organized system of stages for explanation and understanding of any problematic situation investigated. The formal component, however, involves axioms, definitions, theorems and demonstrations and is manifested in that the abstraction establishes itself and needs a more symbolic language to represent advanced mathematical thinking.

According to the arguments of Fischbein (1987) about the connections between the components of a mathematical activity, there is a wealth of possibilities to increase the students' ability to imagine and understand the conceptual aspects drawn from problematic situations investigated, resulting in the increase of mathematical creativity and imagination.

This way of developing the investigatory activity in mathematics education fosters the development of inquisitive thinking in the students, leading them to a practice of reality interpretation. This process of mathematical reading of the world might help students to discuss their ideas across the school environs and even beyond it, regardless of the material conditions which it possesses. The necessary ingredient for this cognitive act is the teacher’s action and the evocative environment which should be established in the classroom. It is essential that the teacher avails himself to learn again his way of teaching mathematics, since the most successful educational proposals currently pervade the need of investigation as an alternative to enable the integral growth of students towards their intellectual maturity and the overcoming of inequalities provided by the lack of knowledge on school and scientific mathematics.

In the preparation of investigative actions in math classes, students will be able to develop their creativity and sense of ownership, so that everyone will tend to take an active role in their own learning, engaging deeply in the formulation of surveyed mathematical ideas, going on to feel like discoverers of each mathematical topic investigated. The sense of ownership refers to the personal skill that every individual has in realizing their way of creating cognitive mechanisms of inquiring themselves, understanding and explaining the processes of how things work.

In addition to these benefits, students will develop their self-confidence in a growing and mature way, especially in times of investigation and socialization of their experiences with other colleagues involved in the investigation process in the classroom. This will involve training students able to encourage themselves in the demonstration of mathematical principles observed during the investigation. This course will certainly emphasize the living and globalizing nature of the mathematics included in the investigation, through connections between mathematics and other academic disciplines which may provide links between the external context and the classroom.

As for the evaluation, it is important to consider that the researching practice in mathematics classes involves both exercise and expression of subjectivity and objectivity of those who practice it. These two aspects will be evident more broadly or not, according to the criteria used by the person who develops and evaluates them. Soon, it falls to the teacher to lead the advice to the students so that all of them go on to identify both the objective and subjective aspects of research in order to establish quantitative parameters and qualitative validation of the results obtained in each step of the investigative exercise in mathematics, as well as an evaluation of their learning process.

To conduct a reflection on the execution of a didactical exercise like this, in the training of teachers who teach mathematics, I point out the need for universities to facilitate the formation of a teacher who develops their abilities to think the school mathematics allied to the social, historical and cultural processes of this knowledge's production, considering that the socio-cultural decontextualization which mathematics has been suffered has not led the school in the training of students with an inquiring or investigating spirit. There is the need, therefore, of a stimulus to the act of learning to learn on which the researching practice shows itself as a guiding principle of the cognitive act of thinking, constituting itself in the formative element of the researcher-teacher with a profile which is more convergent to the act of doing in order to learn.

The use of researching projects in education is a possible alternative to the training of teachers who teach mathematics, due to supporting among teachers and students an interactive relationship in the construction of school mathematics, considering mathematics as a human production. This methodological possibility highlights the importance of research as a way of conducting the teacher in training on the (re)elaboration of existing knowledge in mathematics textbooks, as well as to develop activities related to investigation in mathematics education.

The practice of mathematical modeling, the investigations in the history of mathematics and ethnomathematics studies may materialize in the form of researching projects of mathematics teaching as a didactic approach to be executed in the classroom throughout the course of training for the teaching of mathematics. These trends in Mathematics Education, when combined with the investigatory skills associated with the use of projects, constitute the basic framework for the organization of a research aiming to verify the origin, development and use of mathematics, i.e., of its formative, informative and utilitarian aspects. Moreover, it contributes heavily to the development of observation, reasoning, method of work, initiative, self-direction, creativity, cooperation, responsibility and self-expression capacities. Its use in the classroom has the merit of acquainting the student with a way of working which they will often find in the practical and current field, in solving community problems.

Some experiences in training teachers who teach mathematics

In the development of an integrated teaching, research and extension project, focused on the initial and continued training of teachers who teach mathematics I have experienced diverse situations which were very meaningful in what concerns the implementation of research as a teaching strategy for mathematics teaching. Most of them relates to the use of research projects in conjunction with ethnomathematics, the history of mathematics or mathematical modeling.

The first experience was developed in initial and continued training courses of teacher who teach mathematics, involving students of degree courses in mathematics and pedagogy of two public universities in the state of Pará, for five years. The experiment focused on the use of research projects in the classroom, where I discussed the importance of developing micro research projects to establish links between classroom mathematics and problem situations encountered in everyday life. In this practice I supervised the participants in the design, development and mathematical modeling of the investigated situations, showing them the ability of supporting the students' learning in Basic Education. The results were so significant that they triggered, among other actions, the elaboration of monographs in the area of undergraduate degree in Mathematics and Education, as well as originated projects of Specialization and Masters degree later.

However, some methodological obstacles in their use were pointed out, such as: 1) Lack of orientation for these students during their initial training at these universities, 2) Difficulties of the students to make connections between the investigated reality and the mathematics that is to be taught in primary education and 3) Difficulties of the students to select research topics which addressed the mathematical contents covered in Basic Education.

Nevertheless, the difficulties mentioned were overcome as I tried to create and propose some directions for overcoming them, for example, film presentations involving practices related to mathematics and technology, the use of magazine and newspaper articles to explore current issues related to mathematics and society, visits to fairs, supermarkets, craft workshops and carpentry, among others. The exercise of overcoming the obstacles highlighted the pedagogical potential of the proposed development (Mendes, 2009a).

The second experience took place during the use of mathematical history as a methodological aid for the teaching and learning of mathematical contents covered in elementary and high school. This experience was developed with teachers who worked in Basic Education in Natal (Rio Grande do Norte) and Belém (Pará), as well as undergraduate students in mathematics from UFRN. In this experiment I sought to highlight the investigative nature on the historical information in mathematics, from which I elaborated and tested educational activities aimed at teaching math to evaluate the possibilities for using them together with the students of these educational levels with the teachers and students of elementary and high school modules.

The results showed that the investigative character of the activities empowers and makes the classes more dynamic, arousing the interest of those who investigate, generating learning. It makes the addressed topics more significant for the learner; more significant because research allows our understanding of the mathematical creation focused on the continual search for answers to human questions, in the various contexts and historical moments, such as the wave motion of mathematical creation to which I referred at the beginning of this conference. Currently, I develop this type of didactical proposal in the training of mathematics teachers at the Federal University of Rio Grande do Norte, as well as in the continuing education of mathematics teachers working in elementary and high school.

The third experiment was done by me and a doctoral student of mine with a mixed group of teachers in training. A study group was organized, composed of undergraduate students in Pedagogy and Mathematics Education from the Federal University of Piauí and two more math teachers from public schools in Teresina (Piauí). With this group five training workshops were held, focused on history and pedagogy of mathematics in an investigative perspective, in order to support the conceptual and didactical training of that group with respect to the mathematical contents addressed in the early years of elementary school. The purpose of the workshops was to develop studies on the history of mathematics which could support the conceptual and didactical formation of a group of undergraduate students in pedagogy and mathematics aiming to elaborate teaching materials and activities based on information drawn from the historical studies which had been undertaken. The material and activities developed would later be used in the continuous training of teachers of Public Schools in Teresina, in the shape of a history workshop of and pedagogy of mathematics, aiming to overcome didactical and conceptual problems arising from their undergraduate training in pedagogy. Based on the information obtained, it was suggested that new procedural referrals on the level of education and university extension which may contribute to the reorientation of initial and continuous training of teachers of the early years, involving the history of mathematics as a didactical and conceptual mediator feature of this training.

Possibilities of a teacher training for research

Given the propositions and arguments which were demonstrated throughout this article, I assure you that to deliver a proposal for initial and continuous training of a teacher-researcher it is necessary to focus on the possibility of using research as a formative principle of this teacher, constantly seeking to build a teaching and learning lively proposal to be used in the classroom at all three levels of education. For this to occur we must have greater understanding of the problems faced in the practices of teachers who teach mathematics and by the undergraduate students in education and mathematics during their initial training. Perhaps then it is possible to elaborate a broader program to use these possibilities in the formation of these licensed professionals.

One way to access this reformulation on the practice of the teacher who teaches math means to establish a dialogue between the trends in Mathematics Education and the specific disciplines of these undergraduate programs (Pedagogy and Mathematics) and then develop investigative studies (researches oriented every semester) articulated to the disciplines of pedagogical training for these teachers such as teaching methods, teaching practices or supervised internships.

The program suggested here should cover mainly the last two years of the training course for the teacher who teaches mathematics. The alliance between the subjects, through research articulated to the trends in Mathematics Education, certainly will favor the formation of a teacher who is more creative and less dependent on textbooks. Also, it will encourage in the undergraduates the investigatory spirit focused on the pursuit of knowledge and production of written text from the investigation.

Under the guidance of teachers of methodology of mathematical teaching and other subjects, students might carry out their studies concerning the socio-historical and cultural aspects of mathematics (or other subjects) focused on the content to be addressed in elementary and high school. From these studies they can build textbooks, concrete materials and activities to be used with students in these grade levels. Such products may promote the development and implementation of small research projects aimed at the teaching of mathematics (or other contents) to be developed throughout the period of supervised practice.

The results obtained may provide the necessary insight for both university professors, and undergraduate students and teachers of elementary and high school may to have a broader view of the process triggered during the study. Thereafter, it becomes possible to discuss strategies for overcoming the difficulties encountered during the teaching practice.

According to the ideas presented in this article, my perspective is made evident in teaching, research and extension to be developed in undergraduate courses in mathematics and pedagogy, as well as in the continuous training of teachers of elementary and high school, considering the need for training a researcher-teacher. It is very important that such studies are carried out by universities and are always articulated with the network of primary and secondary education, for it is from this articulation that a dialogue will arise, in which researchers in Mathematics Education might find an echo to their ideas and certainly will be able to continuously expand their range of coverage in the elaboration of studies and programs which might contribute to overcoming the difficulties encountered by the whole community, when it comes to education.

Finally, I can ensure that the learning and practice of research in teaching and learning of mathematics become essential due to the fact that usually a discovery sheds new light on a multitude of other facts and, therefore, it does not mean one discovery, but several findings. Moreover, as the regions to exploit mathematics enlarge the border between the known and unknown, they offer a more abundant field for investigations and implications of the results for the teaching of mathematics.

For the development of research in the classroom to become effective in a profitable way it is necessary that all students exercise the bibliographical, documentary and experimental research. For that they should learn to consult Internet sources, museum and library archives as well as videographic sources, in addition to observing the reality around them so as to generate problems and inquiries that lead them to lead in finding answers to their questions. These attitudes are indispensable in the conceptual construction of any mathematical topic to be learned from the investigation, because all these sources, when properly exploited, are enriching the process of acquisition of mathematical knowledge in school. From this practice the student may develop a new attitude towards the construction of their mathematical knowledge.

Given all the above, I defend the possibility and appreciation of the cognitive attitude of research in mathematics classes, provided there is a continuous search for ways of enriching the reformulations of the educational planning of the teacher every semester, taking into account that the dynamics of action-reflection must always be present in our practices. It is in search of renewal and reconstruction that we can promote a full learning of mathematics.

An attempt to overcome the difficulties is the reformulation of the practice of the mathematics teacher. This will be possible with the development of a joint project among all the mathematics teachers in the school, so as to achieve a collective work or even to establish a dialogue between the subject of mathematics and the other subjects, in order to achieve a more interdisciplinary approach in the mathematics training of the students, especially in the exercise of research in the classroom.

Taking the sociocultural and historical aspects related to mathematics as a guide of the investigation, students can do several researches involving related topics across the various subjects addressed during the school year in order to facilitate its constructive learning of the contents to be taught by teachers in Middle or Elementary School. From these studies, teachers will be able to organize, with the students, texts which address mathematical topics with more comprehensive features which point out the crosscutting nature of mathematics in various fields of knowledge.

The results obtained may contribute so that both the teachers and the students of these educational levels obtain a broad overview of the mathematical production process and its implications in the context of society and culture. Thereafter, it will be possible to explore the various strategies for overcoming the difficulties encountered during the teaching practice, provided that the teacher always have a guideline of the investigative work involving mathematics in the classroom.

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